

THE INNER STRUCTURE OF HYBRID STARS

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Abstract Hybrid stars with extremely high central energy density in their core are natural laboratories to investigate the appearance and the properties of compactified extra dimensions with small compactification radius – if these extra dimensions exist at all. We introduce the necessary formalism to describe quantitatively these objects and the properties of the formed hydrostatic equilibrium. Different scenarios of the extra dimensions are discussed and the characteristic features of these hybrid stars are calculated.

Keywords: Neutron stars, quark stars, compactified extra dimensions.

Introduction

Neutron and quark stars are natural laboratories to investigate the interplay of strong, electro-weak and gravitational interaction. Many theoretically determined properties of these astrophysical objects were tested by the observed properties of pulsars, and detailed calculations exist for these stars[1–4].

However, if new perspectives appear in the description and understanding of the gravitational interaction or in the unification of the above interactions, then revisiting of the models becomes necessary. Such a reinvestigation was triggered by the refreshed attention on compactified extra dimensions[5]. Extra dimensions inside neutron stars were investigated earlier[6], but the Kaluza-Klein (K-K) excitation modes were not considered in the equation of state (EoS). These modes are important constituents of the recent gravitation theories. Introducing the K-K modes into the EoS of fermion stars at their central core, new features and properties emerged [7].

Here we display a few of our ideas about these extra dimensions, their possible connection to particle physics and their appearance in the core of hybrid stars. We summarize our numerical results and discuss the observability of extra dimensions in these objects.

1. The Fifth Dimension

The introduction of the 5th dimension into the real World has a long history. We do not have any direct information about the extra dimensions, so we have an alternative. Either x^5 does not exist, or, it is microscopically small and compact. Obviously in the present paper we take the second horn of the alternative, for details see Refs. [8, 9].

Quantization puts a serious constraint on five-dimensional motion. If there is an independence on x^5 , then the particle is freely moving in x^5 . However, being that direction compactified leads to an uncertainty in the position with the size of $2\pi R_c$, where R_c is the compactification radius. Thus a Bohr-type quantum condition appears formally:

$$p^5 = \frac{n\hbar}{R_c} . \quad (1)$$

Because of the extra motion into the fifth dimension, an extra mass term appears in 4D descriptions. Considering compactified radius $R_c \sim 10^{-12} - 10^{-13}$ cm this extra "mass" is $\hat{m} \sim 100$ MeV; similar values appear in Ref. [5]. A quite recent approach of hadron spectra by Arkhipov [11] is also worthwhile to consult with.

An interesting consequence of the existence of 5th dimension is that in the "4 dimensional" observations an apparent violation of the equivalence principle must appear as one can show it by writing the geodesic equation in 5 dimensions and then projecting it into 4 dimensions. Without going into details, the \pm sign of p^5 causes the appearance of a "pseudo-charge" \hat{q} in the 4 dimensional formalism,

$$\hat{q} = n \cdot \frac{2\hbar\sqrt{G}}{c R_c} , \quad (2)$$

which acts in a vector-scalar interaction. We can directly see that \hat{q} is *not* the electric charge. Indeed $\hat{q}^2 < 16\pi G m_0^2$, where G is the gravitational constant and m_0 is the rest mass [10]. This is either some familiar quantum number (e.g. strangeness, S), or some other not yet observed charge.

2. Field Equations

We are interested in final states of stellar evolution. Therefore we can restrict ourselves to static configurations. Also, fluid-like behavior seems appropriate *in the microscopical dimensions*. Therefore we are looking for *static* configurations. Also, some fluid-like behaviour is expected in the sense that stresses in the *macroscopical* directions freely equilibrate. Then in 3 spatial directions *isotropy* is expected and thence *spherical symmetry*. Finally, in the lack of any information so far, we may assume symmetry in the extra dimension. Then in

proper GR language we are looking for solutions with 5 Killing vectors

$$K_{Ai;k} + K_{Ak;i} = 0$$

$A = 0, 1, 2, 3, 5$, with the commutation relations

$$[K_0, K_B] = 0, \quad [K_5, K_B] = 0, \quad [K_\alpha, K_\beta] = \epsilon_{\alpha\beta\gamma} K_\gamma \quad (3)$$

so that the subgroup 1, 2, 3 acts with 2 dimensional transitivity. Hence the metric has the unique form

$$ds^2 = e^{2\nu} dt^2 - e^{2\lambda} dr^2 - r^2 d\Omega^2 - e^{2\Phi} (dx^5)^2, \quad (4)$$

where the quantities ν , λ and Φ depend only on r . As for the energy-momentum tensor we get

$$T^{ik} = \text{diag} (\varepsilon e^{2\nu}, P e^{2\lambda}, P r^2, P r^2 \sin^2 \theta, P_5 e^{2\Phi}). \quad (5)$$

Total spatial equipartition of particle momenta is not expected (so $P \neq P_5$); indeed we shall see that it does not happen generally.

The Einstein equations are read as

$$-\gamma \varepsilon = e^{-2\lambda} \left[\Phi'' + \Phi'^2 - \lambda' \Phi' + \frac{2\Phi'}{r} - \frac{2\lambda'}{r} + \frac{1}{r^2} \right] - \frac{1}{r^2} \quad (6)$$

$$-\gamma P = e^{-2\lambda} \left[-\nu' \Phi' - \frac{2\Phi'}{r} - \frac{2\nu'}{r} - \frac{1}{r^2} \right] + \frac{1}{r^2} \quad (7)$$

$$-\gamma P = e^{-2\lambda} \left[-\nu'' - \nu'^2 + \nu' \lambda' - \Phi'' - \Phi'^2 - \nu' \Phi' + \lambda' \Phi' - \frac{2\Phi'}{r} - \frac{\nu'}{r} + \frac{2\lambda'}{r} \right] \quad (8)$$

$$-\gamma P_5 = e^{-2\lambda} \left[-\nu'' - \nu'^2 + \nu' \lambda' - \frac{2\nu'}{r} + \frac{2\lambda'}{r} - \frac{1}{r^2} \right] + \frac{1}{r^2} \quad (9)$$

where $\gamma = 8\pi G/c^4$ and all quantities can depend only on the radius r .

For final states ($T = 0$) all local material characteristics of the fluid are expected to depend on one thermodynamic quantity, say on particle density n . The material equations,

$$\varepsilon = \varepsilon(n); P = P(n); P_5 = P_5(n), \quad (10)$$

and the Einstein equations in eqs. (6)-(9) are all independent equations, all further equations are consequences. (Indeed there are some thermodynamical constraints between ε, P and P_5 .) Considering the appropriate Bianchi identity, one obtains

$$T^{ir}_{;r} = 0 \longrightarrow P' = -\nu'(\varepsilon + P) + (P_5 - P)\Phi'. \quad (11)$$

This equation clearly demonstrates the influence of the extra dimensional behavior on the normal pressure, $P = P_1 = P_2 = P_3$.

The Einstein equations eqs. (6)-(9) contain two extra variables compared to the more familiar 4 dimensional case, namely P_5 and Φ . However, $P_5(n)$ is a known function of the particle number density and specified by the actual interaction in the matter. Thus $\Phi(r)$ is the only new degree of freedom determined by the extra equation.

3. A Special Solution of the Field Equations

For specially chosen pressure P_5 there is a unique solution of the Einstein equations in eqs. (6)-(9), namely $\Phi' = 0$. In this case eqs. (6)-(8) can be solved separately with $\Phi = 0$ and then the last eq. (9) gives P_5 . Although these solutions do not differ formally from the 4 dimensional neutron star solution (except for P_5), but the extra dimension will have its influence on $\varepsilon(n)$ and $P(n)$ [7].

Let us start with a single massive fermion ("neutron", N). Since the minimal nonzero fifth momentum component is $|p^5| = \hbar/R_c$, then the extra direction of the phase space is not populated until the Fermi-momentum $p_F < \hbar/R_c$. However, at the threshold both $p^5 = \pm \hbar/R_c$ states appear. They mimic another ("excited") particle with mass $m_X = \sqrt{m_N^2 + (\hbar/R_c)^2}$ (with a nonelectric "charge" $\hat{q} = \pm \frac{2\hbar\sqrt{G}}{cR_c}$ as well). The equations obtain a form as if this second particle also appeared in complete thermodynamical equilibrium with the neutron: $\mu_X = \mu_N$. This phenomena is repeated in any case, when p exceeds a threshold $n\hbar/R_c$.

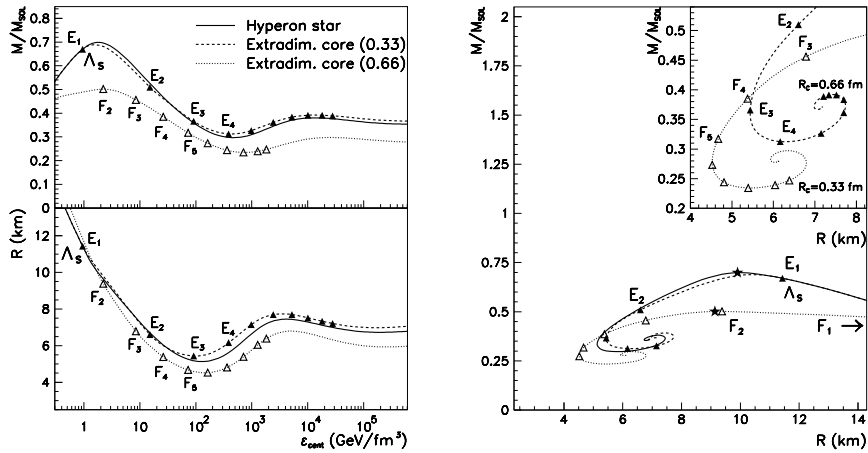


Figure 1. The mass and the radius of heavy hadron stars in 4 dimensions and neutron stars including extradimensional K-K modes into the core (from Ref. [9]).

Fig. 1. compares two calculations. The solid line is 4 dimensional calculation with neutrons and Λ hyperons, the lightest neutral strange baryon. (Ambartsumyan and Saakyan calculated such hybrid compact stars as far back as in 1960 [12].) The dashed and dotted lines come from 5-dimensional calculations with only neutrons, but moving in the extra dimension, too. The higher excitations are started from the triangles and they are named as E_1, E_2, \dots , at compactified radius $R_c = 0.33$ fm. In this case the 5 dimensional neutron star is almost indistinguishable from a neutron star with Λ core. However, choosing another compactification radius (e.g. $R_c = 0.66$ fm) one obtains reasonable differences, which can be seen on Fig. 1. The F_2, F_3, \dots , indicate the appearance of the excited modes in the latter case (F_1 was left outside of the figure).

4. On the General Solution of the Field Equations

In the most generic case it is true that the derivative $\Phi' \neq 0$. The interior solution of eqs. (6)-(9) can be obtained only numerically. In principle there is no problem to evaluate these equations, but the structure of them must be understood.

First, there are constraints among the material quantities, which are rather straightforward for ideal quantum fluids. In case of interacting matter one can recall the self-consistent description of the interacting fermions in the 4 dimensional analogy [13]. Second, the particle number density in the center, $n(r=0)$, is a free initial condition, as it was in the 4 dimensional case. Instead of density, we may use energy density, $\varepsilon(r=0) \equiv \varepsilon_0$, as well.

Third, the metric tensor is determined by the variables Φ, ν, λ . On the other hand Φ and ν never appear in eqs.(6)-(9) (reflecting the fact that x^0 and x^5 constant dilatations are always possible without harming the commutator relations for the Killing motions), so these equations are of first order on Φ', ν' and λ . However, the equations can be rearranged resulting in the following symbolic structure:

$$\Phi' = \Phi'(\lambda', \lambda, n) \quad (12)$$

$$\nu' = \nu'(\lambda', \lambda, n) \quad (13)$$

$$\lambda'' = \lambda''(\lambda', \lambda, n) \quad (14)$$

The eqs. (12)-(13) are algebraic equations for Φ' and ν' , but eq. (14) is a differential equation of second order and it should be reorganized formally into 2 equations of first order.

Fourth, initial conditions are needed for Φ, ν, λ and λ' . For the first two they are free additive constants and for λ and λ' we can proceed as in 4 dimensions. Anyway, it is clear that even if we choose such initial conditions that $\Phi'(r=0) = 0$, then it does not remain zero out of the center.

5. Matching Conditions on the Surface

The integration must go until the fluid-vacuum interface at r_s , where an exterior vacuum solution continues the interior one. The details of matching conditions can be found in Ref. [14]. Applying them on the present problem we get the continuity of certain derivatives of the metric tensor, and those for the energy-momentum tensor result in the single equation,

$$P(r_s) = 0 . \quad (15)$$

Observe that no constraint is obtained either for ε or P_5 .

Although P_5 might be anything at r_s from the matching conditions, it will be zero for all general (not very exotic) systems. Namely, $P = 0$ is expected at some *low* particle number density n_s on the surface, which is generally much below $p_5 = \hbar/R_c$. So motions in the 5^{th} dimension cease already somewhere in the interior of the star.

Henceforth vacuum solutions of eqs. (6)-(9) are valid until infinity, and $\Phi(r_s)$, $\nu(r_s)$, $\lambda(r_s)$, and $\lambda'(r_s)$ give initial conditions for the external solution. Interestingly enough, the external solutions can be obtained in analytical (albeit partly implicit) forms.

6. The External Solution

The external solution of the Einstein equations can be obtained in analytic form. Equating the matter contribution with zero on the left hand sides and using the new variables α and β as

$$\nu' \equiv \alpha + \beta \quad (16)$$

$$\Phi' \equiv \alpha - \beta , \quad (17)$$

the 4 Einstein equations results in 3 differential equations,

$$\lambda' = \frac{1}{r}(1 - e^{2\lambda}) + 2\alpha \quad (18)$$

$$\psi' = -\frac{\psi}{r^2}(1 + e^{2\lambda}) , \quad (19)$$

(where $\psi \equiv \alpha, \beta$), and one more equation becomes an identity. Eqs. (18)-(19) can be integrated as

$$\alpha = \frac{A}{r}e^{-Y} \quad (20)$$

$$\beta = \frac{B}{r}e^{-Y} \quad (21)$$

with two constants A, B . For variable Y eq.(18) yields the following:

$$Y' = \frac{1}{r} \left[(A^2 + B^2)e^{-2Y} + 4Ae^{-Y} + 1 \right] \quad (22)$$

This last equation can be solved in an implicit form. There are three solutions according to the sign of the determinant. Here we give the one for positive determinant:

$$\ln \left[\frac{r}{r_0} \right] = \ln \left[\left(\frac{y - y_+}{y_0 - y_+} \right)^u \left(\frac{y - y_-}{y_0 - y_-} \right)^v \right] \quad (23)$$

where r_0 and y_0 are constants of integration. Furthermore,

$$D = \sqrt{3A^2 - B^2} \quad (24)$$

$$y_{\pm} = -2A \pm D \quad (25)$$

$$u = \frac{-2A + D}{2D} \quad (26)$$

$$v = \frac{+2A + D}{2D} \quad (27)$$

and then

$$Y = \ln y \quad (28)$$

$$e^{2\lambda} = (A^2 + B^2)e^{-2Y} + 4Ae^{-Y} + 1 \quad (29)$$

Of course, this solution goes to flat space-time as $r \rightarrow \infty$.

The $\Phi' = 0$ special solution belongs to $A = B$. However, as we claimed earlier, $\Phi'(r_s) \neq 0$.

7. Conclusion

We do not know compact stars close to Sun, so the observations are rather hazy for details. However some data exist for masses and radii. In the previous Chapters we showed that the calculations are as straightforward in 5 dimensions as in 4. While some problems must be relegated to the next, last Chapter, in principle one could compare observations with 4- and 5-dimensional calculations, and try to decide if space is 3 dimensional or 4. The difference comes from two reasons.

1) At some Fermi momentum the phase space opens up in the 5th direction. Henceforth p_F increases slower, which will be reflected by the $M(R)$ relation. However note that 5th dimensional effects can mimic particle excitations.

2) Quite new effects can, in principle, be observed around compact stars. Of course, they may be moderate and the present observational techniques are probably insufficient. However, consider 2 situations:

You can perform light deflection experiments. Trajectories differ in 4 and 5 dimensions.

You may perform "free fall" experiments in vacuum but not very far from the star. Then 1) particles moving also into the extra direction would fall "anomalously", but this is a simple Eotvos-type experiment and can be done in lab too.

However 2) $\Phi' \neq 0$, so also a "scalar force" appears for particles moving in the extra dimension, although we are outside of the exotic interior of the star. This means that accelerator experiments not too far from compact stars would show up surprising results *if* 5th Dimension exists. So not only the dense source would be different. Gravity is a phenomenon of Space-time and 5-dimensional Space-Time differs from the 4-dimensional one.

Before truly realistic 5-dimensional calculations some problems must be fully clarified. They are listed in the last Chapter.

8. Outlook

While we can calculate 5-dimensional compact stars in this moment, the results still depend on parameters to be fixed in the future. We, somewhat arbitrarily, classify problems into 3 groups: factual, technical and fundamental.

1) Factual problems:

The value of the compactification radius, R_c : In the present approach this radius was a free parameter. For demonstration we chose the radius $R_c = 0.33 \cdot 10^{-13}$ cm, when the strange Λ baryon could behave as the first excitation of a neutron. Such an extradimensional object can mimics a compact star with neutrons in the mantle and Λ 's in the core. With smaller R_c the "exotic" component appears at larger densities – we may run into the unstable region of the hybrid star and the extra dimension remains undetectable. However, with larger R_c the mass gap becomes smaller and the "transition" happens at "familiar" neutron star densities. In this way, reliable observations could lead to an upper bound on R_c .

The quantum number connected with motion in x^5 : We saw that a neutron, starting to move in the 5th direction, appears in macroscopic 4-dimensional description as if a neutral baryon, but with a higher mass. Particle tables mention such particles in abundance. But we also mentioned that 5 dimensional GR gives that the 4 dimensional projection of the geodesics of such particles would exhibit a really weak apparent violation of the equivalence principle, via a vector-scalar force. In first approximation this would mimic a Coulomb-like force, but cca. 10^{-39} times weaker. Now, there was one serious attempt to see the violation of the equivalence principle (in the Eotvos experiment [15]); there it seemed that the strength was in this range. The tentative idea was that the violation might be connected with hypercharge $Y = B + S$, and there was also a discussion [16] whether the mass difference of K^0 and anti- K^0 came from the interaction of Earth's particles with the kaon in lab. Now, in some sense the "first excitation" of n in Y is Λ . It would be a nice minimal theory if Fifth Dimension would explain the CP-violation of weak interaction as well, but of course we cannot expect this.

The way of compactification: Here we chose the simplest compactification: $x^5 \equiv x^5 + 2\pi R_c$. This is a cylindrical compactification. However more complicated cases are also possible, albeit not arbitrary ones. A branch of GR lists all the cases; now we mention only the 2-dimensional example that from a plane by cut and sew you can produce of course a compact cylinder, but the mantle of a cone as well.

The number of the extra dimensions: The present example was the simplest nontrivial case. However maybe particle physicists would prefer 6 extra dimensions appearing with the same length scale in some group structure. The "extra forces" appearing from Fifth Dimension are gravitational, the extra quantum number appearing here cannot be the source of QCD forces [10]. In Ref. [9] we discussed *two* extra dimensions with different scales.

2) Technical problems:

Initial conditions: We mentioned that we need to fix 3 initial conditions in the center: one for the central density of matter ($n(0) = n_0$ or $\epsilon(0) = \epsilon_0$), and two for the metric, either for $\lambda(0)$ and $\lambda'(0)$, or, equivalently, for $\lambda(0)$ and $\Phi'(0)$. But we cannot properly impose these conditions in $r = 0$, and these conditions are somehow not independent. However the technical problem is well known already in 4 dimensions [1]. First, the proper way is to approximate the innermost core of radius δ with a homogeneous sphere of density n_0 , where the exact value of δ is irrelevant if small enough. Then $n = n_0$ at $r = \delta$, and $e^{2\lambda} = 1 - 8\pi\delta^3\epsilon_0/3c^2$ there.

Interactions: $N - N$ interaction can be taken into account the same way as in Ref.[13]. Neutrons moving into the extra dimension may interact differently because of an extra momentum dependence. Such momentum dependence appears e.g. in Walecka-type [17, 18] construction.

3) Theory:

A consequent 5-dimensional treatment would require Unified Theory of Quantum Mechanics and General Relativity. This unified theory is not available now, and we know evidences that present QM is incompatible with present GR. The well-known demonstrative examples are generally between QFT and GR (e.g. the notion of Quantum Field Theory vacua is only Lorentz-invariant and hence come ambiguities about the existence of cosmological Hawking radiations [19]). But also, it is a fundamental problem that the lhs. of Einstein equation is c -number, while the rhs. should be a quantum object.

In the present case QM and GR have to be used in a compatible way for the proper "Bohr-type" quantization of p^5 . Our chosen way was intuitive. Sure, it is correct for $\Phi' = 0$. Then one can always introduce such a coordinate system where $\Phi = 0$ and then the circumference is the usual $2\pi R_c$. But in the general case one might use $2\pi R_c$ in the condition as well as $2\pi e^\Phi R_c$. All suggestions up to now (as. e.g. Ref. [20]) are restricted to the simpler case.

This paper has displayed the introduction of 5th dimension into astrophysical description in the framework of GR. The new degrees of freedom may solve old problems, but open new questions, as well. We think that this attempt is worthwhile.

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